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Equivariant Brauer groups and cohomology ☆

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Abstract

In this paper we present a cohomological description of the equivariant Brauer group relative to a Galois finite extension of fields endowed with the action of a group of operators. This description is a natural generalization of the classic Brauer–Hasse–Noether's theorem, and it is established by means of a three-term exact sequence linking the relative equivariant Brauer group, the 2nd cohomology group of the semidirect product of the Galois group of the extension by the group of operators and the 2nd cohomology group of the group of operators.

Keywords: Cohomology; Brauer group; Central simple algebra; Galois extension; Ring with operators; Group of operators

0. Introduction

In algebra, one of the most classic topics related to the 2nd cohomology group of a group deals with the study of Brauer groups. Since the 1930's, it has been known that, for any Galois finite field extension E/K with Galois group G, there is an isomorphism $H^2(G, E^\times) \cong \mathrm{Br}(E/K)$, between the 2nd cohomology group of G with coefficients in the G-module $E^\times$, the multiplicative group of E, and the relative Brauer group of the exten-

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sion $\mathrm{Br}(E/K) = \mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(E))$, that is, the subgroup of the Brauer group of K consisting of those equivalence classes of (finite-dimensional) central simple K-algebras that are split by E. This relative Brauer group can also be described as the group of isomorphism classes of central simple K-algebras containing E as a maximal commutative subalgebra.

This paper deals with a suitable counterpart, in an equivariant situation, of the aforementioned classic Brauer–Hasse–Noether result.

If Γ is a group acting, by automorphisms, on a field K, then the equivariant Brauer group $\mathrm{Br}(K,\Gamma)$ by Fröhlich and Wall [4–6], consists of equivariant Morita-equivalence classes of central simple (K,Γ)-algebras, that is, of central simple K-algebras endowed with a Γ-action by ring automorphisms extending the given Γ-action on K. As in the ordinary case, if E/K is any extension of Γ-fields, then the relative equivariant Brauer group $\mathrm{Br}(E/K,\Gamma)$ is the kernel of the induced homomorphism $\mathrm{Br}(K,\Gamma) \to \mathrm{Br}(E,\Gamma)$, and the main purpose of this article is to prove the following:

Theorem. *Let Γ be a group and let E/K be a finite-dimensional Galois extension of Γ-fields with Galois group G. Then there is a natural exact sequence*

$$0 \to \mathrm{Br}(E/K,\Gamma) \to H^2\big(G \rtimes \Gamma, E^\times\big) \to H^2\big(\Gamma, E^\times\big), \tag{1}$$

where $G \rtimes \Gamma$ is the semidirect product group associated to the diagonal Γ-action on G.

This theorem has a precedent in [2, Theorem 1], in the case where Γ acts trivially on K. The authors showed a cohomological description of the group, denoted by $\mathrm{Br}_\Gamma(E/K)$, of equivariant isomorphism classes of central simple K-algebras endowed with a Γ-action by K-automorphisms and containing E as a Γ-equivariant strictly maximal subfield. However, a cohomological description for $\mathrm{Br}_\Gamma(E/K)$ such as this one uses equivariant group cohomology instead of ordinary group cohomology and, moreover, the group $\mathrm{Br}_\Gamma(E/K)$ is not isomorphic to the equivariant Brauer group $\mathrm{Br}(E/K,\Gamma)$ of Fröhlich and Wall considered here. Indeed, the group $\mathrm{Br}(E/K,\Gamma)$ is a nontrivial quotient of the group $\mathrm{Br}_\Gamma(E/K)$, which must be stressed due to its unusual nature (as opposed to the nonequivariant case).

We prove the existence of the exact sequence (1) in the theorem by using certain manageable cochain complexes $\bar{\mathcal{C}}(G \rtimes \Gamma; M)$, associated to Γ-groups G and $(G \rtimes \Gamma)$-modules M, that allow us to compute the cohomology groups of semidirect product groups $G \rtimes \Gamma$ by means of "reduced cocycles." This justifies Section 1 of the paper, where we introduce these complexes and prove that they are a "reduction" of the ordinary complexes $\mathcal{C}(G \rtimes \Gamma; M)$, in the sense that there are quasi-isomorphisms

$$\bar{\mathcal{C}}(G \rtimes \Gamma; M) \to \mathcal{C}(G \rtimes \Gamma; M)$$

and each abelian group of "reduced" n-cochains $\bar{\mathcal{C}}^n(G \rtimes \Gamma; M)$ is smaller than the ordinary $\mathcal{C}^n(G \rtimes \Gamma; M)$. Moreover, we believe that these complexes $\bar{\mathcal{C}}(G \rtimes \Gamma; M)$ may be of interest separately from their usefulness herein.

In Section 2 we briefly review some basic facts, concepts and terminology concerning relative equivariant Brauer group theory for fields. The material in this section is the usual for the nonequivariant case (see [8], for example). The arguments for the proofs can be summarized by noting that the normal proofs work as usual and have consequently been skipped here. The next section is dedicated to proving the above stated theorem and includes other facts and consequences of interest to the reader.

1. On the cohomology of a semidirect product group

To fix some notations, let us briefly recall that, for any group G and any G-module M, the cohomology groups $H^n(G;M)$, [1], can be computed as the homology groups of the cochain complex $\mathcal{C}(G;M)$, in which each $\mathcal{C}^n(G;M)$ consists of all maps $\varphi: G^n \to M$ such that $\varphi(x_1,\dots,x_n)=0$ whenever $x_i=1$ for some $i=1,\dots,n$, and the coboundary $\partial: \mathcal{C}^{n-1}(G;M) \to \mathcal{C}^n(G;M)$ is defined by

$$
\begin{aligned}
&(\partial\varphi)(x_1,\dots,x_n)\\
&\quad = {}^{x_1}\varphi(x_2,\dots,x_n) + \sum_{i=1}^{n-1}(-1)^i\varphi(x_1,\dots,x_ix_{i+1},\dots,x_n) + (-1)^n\varphi(x_1,\dots,x_{n-1}).
\end{aligned}
$$

If $G' \leqslant G$ is a subgroup of G, then there is an induced epimorphism of complexes, $\mathrm{res}: \mathcal{C}(G;M) \to \mathcal{C}(G';M)$, whose kernel is denoted by $\mathcal{C}(G,G';M)$. Thus, we have a short exact sequence of complexes

$$0 \to \mathcal{C}(G,G';M) \xrightarrow{\text{in}} \mathcal{C}(G;M) \xrightarrow{\text{res}} \mathcal{C}(G';M) \to 0.$$

The homology of $\mathcal{C}(G,G';M)$ is the cohomology of G with coefficients in the G-module M relative to G', and it is denoted by $H^n(G,G';M)$, $n \geqslant 0$. Then, the long exact sequence in cohomology deduced from the above short exact sequence of cochain complexes has the usual form:

$$\cdots \to H^n(G,G';M) \xrightarrow{\text{in}} H^n(G;M) \xrightarrow{\text{res}} H^n(G';M) \xrightarrow{\delta} H^{n+1}(G,G';M) \to \cdots. \tag{2}$$

Suppose now that Γ is any fixed group (of operators) and let G be a Γ-group, that is, G is a group equipped with an action of Γ by a homomorphism $\phi: \Gamma \to \mathrm{Aut}(G)$. As usual, we denote ${}^{\sigma}x$ by $\phi(\sigma)(x)$, for any $\sigma \in \Gamma$ and $x \in G$, and we denote with $G \rtimes \Gamma$ the corresponding semidirect product group. That is to say, $G \rtimes \Gamma$ is the $G \times \Gamma$ set with multiplication $(x,\sigma)(y,\tau) = (x\,{}^{\sigma}y, \sigma\tau)$. We shall identify both G and Γ as subgroups of $G \rtimes \Gamma$ by means of the obvious injections $x \mapsto (x,1)$, $x \in G$, and $\sigma \mapsto (1,\sigma)$, $\sigma \in \Gamma$.

The main objective of this section is to introduce a certain cochain complex $\bar{\mathcal{C}}(G \rtimes \Gamma;M)$, associated to any Γ-group G and any $(G \rtimes \Gamma)$-module M, and then to prove that this complex is a reduction of the ordinary complex $\mathcal{C}(G \rtimes \Gamma;M)$, in the sense that there is an inducing homology isomorphism complex homomorphism (i.e., a quasi-isomorphism of complexes)

$$\Phi: \bar{\mathcal{C}}(G \rtimes \Gamma;M) \to \mathcal{C}(G \rtimes \Gamma;M),$$

where each abelian group of n-cochains $\bar{\mathcal{C}}^n(G \rtimes \Gamma; M)$ is smaller than the corresponding $\mathcal{C}^n(G \rtimes \Gamma; M)$.

Before introducing the complex $\bar{\mathcal{C}}(G \rtimes \Gamma; M)$, we require the following easy observation.

Lemma 1.1. *Let G be a Γ-group. A $(G \rtimes \Gamma)$-module is the same as an abelian group M, which is both a Γ-module and a G-module such that, for each $\sigma \in \Gamma$, $x \in G$ and $a \in M$, the following equality holds*:

$$^{\sigma}\big({}^{x}a\big) = {}^{(^{\sigma}x)}\big({}^{\sigma}a\big). \tag{3}$$

Proof. It is straightforward, since a $(G \rtimes \Gamma)$-action on an abelian group M is determined, by the induced Γ-action and G-action, by the equalities ${}^{(x,\sigma)}a = {}^{x}({}^{\sigma}a)$. □

Definition 1.2. Let G be a Γ-group and let M be a $(G \rtimes \Gamma)$-module. The complex of reduced cochains of $G \rtimes \Gamma$ with coefficients in M, $\bar{\mathcal{C}}(G \rtimes \Gamma; M)$, is defined as follows: the nth cochain group $\bar{\mathcal{C}}^n(G \rtimes \Gamma; M)$ is the abelian group of all maps

$$\varphi : \bigcup_{p+q=n} G^p \times \Gamma^q \to M, \tag{4}$$

which are normalized in the sense that $\varphi(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q) = 0$ whenever $x_i = 1$ or $\sigma_j = 1$ for some $i = 1, \ldots, p$ or $j = 1, \ldots, q$. Addition in $\bar{\mathcal{C}}^n(G \rtimes \Gamma; M)$ is given by adding pointwise in the abelian group M. The coboundary

$$\partial : \bar{\mathcal{C}}^{n-1}(G \rtimes \Gamma; M) \to \bar{\mathcal{C}}^n(G \rtimes \Gamma; M)$$

is defined by the formula ($n \geqslant 1$):

$$\begin{aligned}
&(\partial\varphi)(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q)\\
&\quad = {}^{\sigma_1}\varphi(x_1, \ldots, x_p, \sigma_2, \ldots, \sigma_q) + \sum_{j=1}^{q-1}(-1)^j \varphi(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_j\sigma_{j+1}, \ldots, \sigma_q)\\
&\qquad + (-1)^q \varphi\big({}^{\sigma_q}x_1, \ldots, {}^{\sigma_q}x_p, \sigma_1, \ldots, \sigma_{q-1}\big)\\
&\qquad + (-1)^q \Bigg[{}^{(^{\sigma_1 \cdots \sigma_q}x_1)}\varphi(x_2, \ldots, x_p, \sigma_1, \ldots, \sigma_q)\\
&\qquad + \sum_{i=1}^{p-1}(-1)^i \varphi(x_1, \ldots, x_i x_{i+1}, \ldots, x_p, \sigma_1, \ldots, \sigma_q)\\
&\qquad + (-1)^p \varphi(x_1, \ldots, x_{p-1}, \sigma_1, \ldots, \sigma_q) \Bigg].
\end{aligned}$$

By using (3) one can check that $\partial\partial = 0$. Furthermore, the construction of the complex $\bar{\mathcal{C}}(G \rtimes \Gamma; M)$ is functorial both in G and M.

We next state and prove our main result in this section.

Theorem 1.3. *For any Γ-group G and any $(G \rtimes \Gamma)$-module M, there is a natural quasi-isomorphism of complexes*

$$\Phi = \Phi_{G;M} : \bar{\mathcal{C}}(G \rtimes \Gamma; M) \to \mathcal{C}(G \rtimes \Gamma; M). \tag{5}$$

Hence, there are natural isomorphisms

$$H^n \bar{\mathcal{C}}(G \rtimes \Gamma; M) \cong H^n(G \rtimes \Gamma; M), \quad n \geqslant 0. \tag{6}$$

Proof. Consider the bisimplicial set $S = S(G, \Gamma)$ whose set of (p,q)-simplices is $S_{p,q} = G^p \times \Gamma^q$, $p, q \geqslant 0$. The horizontal face and degeneracy maps are defined by those of the minimal Eilenberg–MacLane simplicial complex $K(G,1)$, namely

$$d_i^h(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q) = \begin{cases} (x_2, \ldots, x_p, \sigma_1, \ldots, \sigma_q), & i = 0, \\ (x_1, \ldots, x_i x_{i+1}, \ldots, x_p, \sigma_1, \ldots, \sigma_q), & 0 < i < p, \\ (x_1, \ldots, x_{p-1}, \sigma_1, \ldots, \sigma_q), & i = p, \end{cases}$$

and the vertical face and degeneracy maps by those of the simplicial complex $K(\Gamma, 1)$ unless $d_q^v : G^p \times \Gamma^q \to G^p \times \Gamma^{q-1}$, which is defined by

$$d_q^v(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q) = \left({}^{\sigma_q}x_1, \ldots, {}^{\sigma_q}x_p, \sigma_1, \ldots, \sigma_{q-1}\right).$$

This bisimplicial set S and the given $(G \rtimes \Gamma)$-module M determine a double cosimplicial abelian group $\mathcal{C}(S; M) = \mathcal{C}^{**}(S; M)$, in which

$$\mathcal{C}^{p,q}(S; M) = \{\varphi : S_{p,q} \to M\}$$

is the abelian group of all maps $\varphi : G^p \times \Gamma^q \to M$, the vertical cofaces $d_v^j : \mathcal{C}^{p,q-1}(S; M) \to \mathcal{C}^{p,q}(S; M)$ are defined by

$$\left(d_v^j \varphi\right)(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q) = \begin{cases} {}^{\sigma_1}\varphi(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q), & j = 0, \\ \varphi(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_j \sigma_{j+1}, \ldots, \sigma_q), & 0 < j < q, \\ \varphi({}^{\sigma_q}x_1, \ldots, {}^{\sigma_q}x_p, \sigma_1, \ldots, \sigma_{q-1}), & j = q, \end{cases}$$

and the horizontal cofaces $d_h^i : \mathcal{C}^{p-1,q}(S; M) \to \mathcal{C}^{p,q}(S; M)$ are defined by

$$\left(d_h^i \varphi\right)(x_1, \ldots, x_p, \sigma_1, \ldots, \sigma_q) = \begin{cases} {}^{({}^{\sigma_1 \cdots \sigma_q}x_1)}\varphi(x_2, \ldots, x_p, \sigma_1, \ldots, \sigma_q), & i = 0, \\ \varphi(x_1, \ldots, x_i x_{i+1}, \ldots, x_p, \sigma_1, \ldots, \sigma_q), & 0 < i < p, \\ \varphi(x_1, \ldots, x_{p-1}, \sigma_1, \ldots, \sigma_q), & i = p. \end{cases}$$

We now denote by $\mathcal{C}_N(S;M)$ the associated Moore bicomplex of normalized cochains; that is, $\mathcal{C}_N(S;M)$ is the bicomplex having as (p,q)-cochains all the maps $\varphi : G^p \times \Gamma^q \to M$ such that $\varphi(x_1,\dots,x_p,\sigma_1,\dots,\sigma_q)=0$ if $x_i=1$ or $\sigma_j=1$ for some i or j, as vertical coboundary

$$\partial_v = \sum_{j=0}^{q}(-1)^j d_v^j : \mathcal{C}_N^{p,q-1}(S;M) \to \mathcal{C}_N^{p,q}(S;M)$$

and as horizontal coboundary

$$\partial_h = \sum_{i=0}^{p}(-1)^i d_h^i : \mathcal{C}_N^{p-1,q}(S;M) \to \mathcal{C}_N^{p,q}(S;M).$$

Then, from a result by Dold and Puppe [3, Theorem 2.15], there is a quasi-isomorphism of cochain complexes

$$f : \operatorname{Tot}\mathcal{C}_N(S;M) \to \operatorname{diag}_N \mathcal{C}(S;M),$$

where $\operatorname{Tot}\mathcal{C}_N(S;M)$ is the total complex of the bicomplex $\mathcal{C}_N(S;M)$ and $\operatorname{diag}_N \mathcal{C}(S;M)$ is the Moore complex of normalized cochains of the cosimplicial abelian group diagonal of the double cosimplicial abelian group $\mathcal{C}(S;M)$. For each $n \geqslant 0$, the map $f : \bigoplus_{p+q=n} \mathcal{C}_N^{p,q}(S;M) \to \mathcal{C}_N^{n,n}(S;M)$ is given by

$$f(\varphi) = d_h^{p+q} \cdots d_h^{p+1} d_v^{p-1} \cdots d_v^1 d_v^0(\varphi) \quad \bigl(\varphi \in \mathcal{C}_N^{p,q}(S;M)\bigr).$$

A straightforward identification shows that $\operatorname{Tot}\mathcal{C}_N(S;M) = \bar{\mathcal{C}}(G \rtimes \Gamma;M)$. Moreover, there is an isomorphism of complexes

$$g : \mathcal{C}(G \rtimes \Gamma;M) \cong \operatorname{diag}_N \mathcal{C}(S;M)$$

given, if $\varphi \in \mathcal{C}^n(G \rtimes \Gamma;M)$, by

$$g(\varphi)(x_1,\dots,x_n,\sigma_1,\dots,\sigma_n) = \varphi\bigl(\bigl({}^{\sigma_1\cdots\sigma_n}x_1,\sigma_1\bigr),\bigl({}^{\sigma_2\cdots\sigma_n}x_2,\sigma_2\bigr),\dots,\bigl({}^{\sigma_n}x_n,\sigma_n\bigr)\bigr),$$

for $(x_1,\dots,x_n,\sigma_1,\dots,\sigma_n) \in S^{n,n} = G^n \times \Gamma^n$. Thus we obtain the required quasi-isomorphism of complexes

$$\Phi = g^{-1} f : \bar{\mathcal{C}}(G \rtimes \Gamma;M) \to \mathcal{C}(G \rtimes \Gamma;M). \qquad \square$$

In order to compute the cohomology of a semidirect product group $G \rtimes \Gamma$ relative to the subgroup Γ, we shall now show a reduction of the relative cochain complex $\mathcal{C}(G \rtimes \Gamma, \Gamma;M)$. First let us observe that, when $G=1$, the trivial group, then $G \rtimes \Gamma = \Gamma$, $\bar{\mathcal{C}}(\Gamma;M) = \mathcal{C}(\Gamma;M)$ and the quasi-isomorphism (5) $\Phi_{1;M} : \bar{\mathcal{C}}(\Gamma;M) \to \mathcal{C}(\Gamma;M)$ is merely the identity map. Then, we define the *relative to Γ complex of reduced cochains of*

$G \rtimes \Gamma$ with coefficients in M, $\bar{\mathcal{C}}(G \rtimes \Gamma, \Gamma; M)$, as the kernel of the induced epimorphism of complexes

$$\mathrm{res}: \bar{\mathcal{C}}(G \rtimes \Gamma; M) \to \bar{\mathcal{C}}(\Gamma; M) = \mathcal{C}(\Gamma; M)$$

that carries the reduced n-cochain $\varphi: \bigcup_{p+q=n} G^p \times \Gamma^q \to M$ to its restriction $\varphi|_{\Gamma^n}: \Gamma^n \to M$.

From the commutative diagram of short exact sequences of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & \bar{\mathcal{C}}(G \rtimes \Gamma, \Gamma; M) \arrow[r, "\mathrm{in}"] \arrow[d, "\Phi"] & \bar{\mathcal{C}}(G \rtimes \Gamma; M) \arrow[r, "\mathrm{res}"] \arrow[d, "\Phi_{G;M}"] & \bar{\mathcal{C}}(\Gamma; M) \arrow[r] \arrow[d, equal, "\Phi_{1;M}"] & 0 \\
0 \arrow[r] & \mathcal{C}(G \rtimes \Gamma, \Gamma; M) \arrow[r, "\mathrm{in}"] & \mathcal{C}(G \rtimes \Gamma; M) \arrow[r, "\mathrm{res}"] & \mathcal{C}(\Gamma; M) \arrow[r] & 0
\end{tikzcd}$$

it follows that the complex homomorphism obtained by restriction of $\Phi_{G;M}$ to kernels

$$\Phi: \bar{\mathcal{C}}(G \rtimes \Gamma, \Gamma; M) \to \mathcal{C}(G \rtimes \Gamma, \Gamma; M) \tag{7}$$

is also a quasi-isomorphism. Therefore, there are natural isomorphisms

$$H^n \bar{\mathcal{C}}(G \rtimes \Gamma, \Gamma; M) \cong H^n(G \rtimes \Gamma, \Gamma; M), \quad n \geqslant 0. \tag{8}$$

At this point we are interested in describing explicitly, in terms of reduced cocycles of $G \rtimes \Gamma$, the five-term exact sequence

$$\begin{tikzcd}
H^1(G \rtimes \Gamma; M) \arrow[r, "\mathrm{res}"] & H^1(\Gamma; M) \arrow[r, "\delta"] & H^2(G \rtimes \Gamma, \Gamma; M) \arrow[dl, "\mathrm{in}"'] \\
& H^2(G \rtimes \Gamma; M) \arrow[r, "\mathrm{res}"] & H^2(\Gamma; M)
\end{tikzcd} \tag{9}$$

which is a portion of the general long exact sequence (2) in the particular case of taking the subgroup $\Gamma \leqslant G \rtimes \Gamma$ and a $(G \rtimes \Gamma)$-module M.

For, we remark that, by using the isomorphism (6), we can describe the cohomology group $H^2(G \rtimes \Gamma; M)$ in terms of reduced 2-cocycles, that is, as the abelian group of cohomology classes of normalized maps

$$\varphi: G^2 \cup G \times \Gamma \cup \Gamma^2 \to M \tag{10}$$

satisfying the cocycle conditions:

$$^x\varphi(y, z) - \varphi(xy, z) + \varphi(x, yz) - \varphi(y, z) = 0, \tag{11}$$

$$\varphi\big({}^\sigma x, {}^\sigma y\big) - {}^\sigma\varphi(x, y) + {}^{\sigma x}\varphi(y, \sigma) - \varphi(xy, \sigma) + \varphi(x, \sigma) = 0, \tag{12}$$

$$ {}^{\sigma}\varphi(x,\tau) - \varphi(x,\sigma\tau) + \varphi\big({}^{\tau}x,\sigma\big) - \varphi(\sigma,\tau) + {}^{({}^{\sigma\tau}x)}\varphi(\sigma,\tau) = 0, \tag{13} $$

$$ {}^{\sigma}\varphi(\tau,\gamma) - \varphi(\sigma\tau,\gamma) + \varphi(\sigma,\tau\gamma) - \varphi(\sigma,\tau) = 0, \tag{14} $$

where two such reduced 2-cocycles $\varphi, \varphi' \in \bar{Z}^2(G \rtimes \Gamma; M)$ are cohomologous if $\varphi' = \varphi + \partial\psi$ for some normalized map

$$ \psi : G \cup \Gamma \to M $$

with

$$ (\partial\psi)(x,y) = {}^{x}\psi(y) - \psi(xy) + \psi(x), \tag{15} $$

$$ (\partial\psi)(x,\sigma) = {}^{\sigma}\psi(x) - \psi\big({}^{\sigma}x\big) + \psi(\sigma) - {}^{({}^{\sigma}x)}\psi(\sigma), \tag{16} $$

$$ (\partial\psi)(\sigma,\tau) = {}^{\sigma}\psi(\tau) - \psi(\sigma\tau) + \psi(\sigma). \tag{17} $$

Similarly, the relative cohomology group $H^2(G \rtimes \Gamma, \Gamma; M)$ can be described, using the isomorphism (8), as the abelian group of cohomology classes of those reduced 2-cocycles $\varphi \in \bar{Z}^2(G \rtimes \Gamma; M)$ as above such that $\varphi|_{\Gamma^2} = 0$, and where two such 2-cocycles $\varphi, \varphi' \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; M)$ define the same cohomology class in $H^2(G \rtimes \Gamma, \Gamma; M)$ if they are made cohomologous as above (i.e., $\varphi' = \varphi + \partial\psi$), but for some normalized map ψ as above such that $\psi|_{\Gamma} = 0$.

Then, the homomorphisms in sequence (9) work, respectively, as follows:

- for $\varphi \in \bar{Z}^1(G \rtimes \Gamma; M)$, $\mathrm{res}([\varphi]) = [\varphi|_{\Gamma}]$,
- for $\varphi \in Z^1(\Gamma; M)$, $\delta([\varphi]) = [\delta_{\varphi}]$, where

$$ \delta_{\varphi}(x,y) = 0 = \delta_{\varphi}(\sigma,\tau) \quad \text{and} \quad \delta_{\varphi}(x,\sigma) = {}^{({}^{\sigma}x)}\varphi(\sigma) - \varphi(\sigma), \tag{18} $$

- for $\varphi \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; M) \subseteq \bar{Z}^2(G \rtimes \Gamma; M)$, $\mathrm{in}([\varphi]) = [\varphi]$,
- for $\varphi \in \bar{Z}^2(G \rtimes \Gamma; M)$, $\mathrm{res}([\varphi]) = [\varphi|_{\Gamma^2}]$.

Several explicit computations of the 2nd cohomology groups of a semidirect product group can be done using reduced 2-cocycles (10). An elementary, although illustrative, example in our context is given below.

Let us consider the Galois extension $\mathbb{C}/\mathbb{R}$, whose Galois group is the cyclic group of order two $G = \langle x \mid x^2 = 1\rangle$, where $x : \mathbb{C} \to \mathbb{C}$ is the complex conjugation automorphism. Let $\Gamma = C_2 = \langle \sigma \mid \sigma^2 = 1\rangle$ be a separated cyclic group of order two acting trivially both on $\mathbb{C}$ and on $\mathbb{R}$. Hence, the induced diagonal action of C_2 on G is also trivial. In this case, a normalized reduced 2-cochain $\varphi \in \bar{\mathcal{C}}^2(G \times C_2, C_2; \mathbb{C}^{\times})$ consists of a pair of complexes $\varphi(x,x), \varphi(x,\sigma) \in \mathbb{C}^{\times}$, since $\varphi(\sigma,\sigma) = 1$. Condition (11), in order for φ to be a 2-cocycle, merely says that $\varphi(x,x) \in \mathbb{R}^{\times}$. Condition (12) says that $\varphi(x,\sigma)$ has a module equal to 1, whereas condition (13) holds if and only if $\varphi(x,\sigma) = \pm 1$. Thus,

$$ \bar{Z}^2\big(G \times C_2, C_2; \mathbb{C}^{\times}\big) = \mathbb{R}^{\times} \times \{\pm 1\}. $$

Since every element in the kernel of $H^2(G \times C_2; \mathbb{C}^\times) \to H^2(C_2; \mathbb{C}^\times)$ is represented by a reduced 2-cocycle $\varphi \in \bar{Z}^2(G \times C_2, C_2; \mathbb{C}^\times)$, and it is plain to see that such a φ is a coboundary in $\bar{C}(G \times C_2; \mathbb{C}^\times)$ if and only if $\varphi(x, x) > 0$ and $\varphi(x, \sigma) = 1$, we conclude that

$$\mathrm{Ker}\big(H^2\big(G \times C_2; \mathbb{C}^\times\big) \to H^2\big(C_2; \mathbb{C}^\times\big)\big) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The exact sequence (1) gives $\mathrm{Br}(\mathbb{C}/\mathbb{R}, C_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, in contrast with the ordinary $\mathrm{Br}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$.

2. Preliminaries on the equivariant Brauer group

The definition, by Fröhlich and Wall, of the equivariant Brauer group is established for any commutative ring on which a group of operators is acting by automorphisms. We only need this notion for a field which we recall below. First, we fix some terminology and notations that will be used hereafter.

In this paper, algebras are always over a field and all algebras and vector spaces are assumed to be of finite dimension. We use the term *Γ-ring* to mean a ring with identity element endowed with a Γ-action by ring automorphisms.

Let K be a Γ-field. A *(K, Γ)-vector space* is a vector space V over K on which a Γ-action by additive automorphisms is given, such that

$${}^\sigma(k \cdot v) = {}^\sigma k \cdot {}^\sigma v,$$

for all $\sigma \in \Gamma$, $k \in K$ and $v \in V$. For two (K, Γ)-vector spaces V, W, the vector space $\mathrm{Hom}_K(V, W)$ may be given the structure of a (K, Γ)-vector space by defining

$${}^\sigma f : v \mapsto {}^\sigma\big(f\big({}^{\sigma^{-1}}v\big)\big) \quad \big(\sigma \in \Gamma,\ f \in \mathrm{Hom}_K(V, W),\ v \in V\big), \tag{19}$$

and the tensor $V \otimes_K W$ is also a (K, Γ)-vector space with the diagonal Γ-action

$${}^\sigma(v \otimes w) = {}^\sigma v \otimes {}^\sigma w \quad (\sigma \in \Gamma,\ v \in V,\ w \in W). \tag{20}$$

A *(K, Γ)-algebra* is a Γ-ring A with a Γ-equivariant ring homomorphism of K into the center of A, so that A is both a K-algebra and a (K, Γ)-vector space. A *homomorphism of (K, Γ)-algebras* is a Γ-equivariant K-algebra homomorphism, that is, a homomorphism of K-algebras $f : A \to B$ such that $f({}^\sigma a) = {}^\sigma f(a)$ holds for any $\sigma \in \Gamma$ and $a \in A$.

If A and B are (K, Γ)-algebras, then the tensor product algebra $A \otimes_K B$ is again a (K, Γ)-algebra with the diagonal Γ-action (20). Moreover, the opposite A^0 of any (K, Γ)-algebra A is again a (K, Γ)-algebra with the same Γ-action of A. Further, if V is any (K, Γ)-vector space, the K-algebra of K-linear endomorphisms of V, $\mathrm{End}_K(V)$, is actually a (K, Γ)-algebra under the diagonal Γ-action (19).

By a *central simple (K, Γ)-algebra*, we mean a (K, Γ)-algebra which is a central simple K-algebra. The following useful lemma is a direct application of the classic Double Centralizer Theorem.

Lemma 2.1. *Let A be a central simple (K,Γ)-algebra, and suppose that $B \subseteq A$ is a central simple (K,Γ)-subalgebra. Then the centralizer of B in A, $C_A(B)$, is a central simple (K,Γ)-subalgebra of A, and there is a Γ-equivariant K-algebra isomorphism $\theta : B \otimes_K C_A(B) \cong A$ given by $\theta(b \otimes c) = bc$.*

For any (K,Γ)-vector space V, $\mathrm{End}_K(V)$ is a central simple (K,Γ)-algebra. Let $\sim_\Gamma$ denote the equivalence relation between central simple (K,Γ)-algebras defined by specifying that $A \sim_\Gamma B$ if and only if there is an isomorphism of (K,Γ)-algebras $A \otimes_K \mathrm{End}_K(V) \cong B \otimes_K \mathrm{End}_K(W)$ for some (K,Γ)-vector spaces V and W.

Definition 2.2 [6, Section 3]. The equivariant Brauer group of a Γ-field K, denoted $\mathrm{Br}(K,\Gamma)$, is the abelian multiplicative group consisting of all $\sim_\Gamma$-classes $[A]$ of central simple (K,Γ)-algebras A. Multiplication is induced by tensor product, that is, $[A][B] = [A \otimes_K B]$. The class $[K]$ is the identity element, and the inverse is given by $[A]^{-1} = [A^0]$.

By [6, Proposition 3.1], we have the following result.

Proposition 2.3. *If A and B are central simple (K,Γ)-algebras, then $A \sim_\Gamma B$ if and only if there is a (K,Γ)-isomorphism $A \otimes_K B^0 \cong \mathrm{End}_K(V)$ for some (K,Γ)-vector space V.*

Later we will need the following lemma, whose proof is, using Lemma 2.1, parallel to the usual one for the ordinary nonequivariant case.

Lemma 2.4. *Let A be a central simple (K,Γ)-algebra. If e is any Γ-invariant nonzero idempotent of A, then eAe is also a central simple (K,Γ)-algebra and $A \sim_\Gamma eAe$, that is, $[A] = [eAe]$ in $\mathrm{Br}(K,\Gamma)$.*

The construction of the equivariant Brauer group is functorial. If E is a Γ-field extension of the Γ-field K, then E is a (K,Γ)-algebra and, for any central simple (K,Γ)-algebra A, the tensor algebra $E \otimes_K A$, endowed with the diagonal Γ-action (20), is a central simple (E,Γ)-algebra. Then, the induced $\mathrm{Br}(K,\Gamma) \to \mathrm{Br}(E,\Gamma)$ is the homomorphism sending $[A]$ to $[E \otimes_K A]$. We say that the Γ-field E *splits* a central simple (K,Γ)-algebra A whenever $[A]$ lies in the kernel of $\mathrm{Br}(K,\Gamma) \to \mathrm{Br}(E,\Gamma)$, that is, if there is an isomorphism of (E,Γ)-algebras $E \otimes_K A \cong \mathrm{End}_E(V)$ for some (E,Γ)-vector space V.

The kernel of $\mathrm{Br}(K,\Gamma) \to \mathrm{Br}(E,\Gamma)$, denoted by $\mathrm{Br}(E/K,\Gamma)$, is the *relative equivariant Brauer group* of the Γ-field extension E/K.

Analogously to the nonequivariant case, we have the following criterion for a Γ-field extension to be a splitting Γ-field.

Proposition 2.5. *Let $\omega \in \mathrm{Br}(K,\Gamma)$ and let E/K be a finite Γ-field extension. Then $\omega \in \mathrm{Br}(E/K,\Gamma)$ if, and only if, for some central simple (K,Γ)-algebra $A \in \omega$ there exists a Γ-equivariant maximal commutative K-algebra embedding $E \hookrightarrow A$.*

3. The relationship between $\mathrm{Br}(E/K,\Gamma)$ and $H^2(G \rtimes \Gamma; E^*)$

Let E/K be a Galois finite field extension with Galois group $G = \mathrm{Aut}_K(E)$. If, in addition, E/K is an extension of Γ-fields, that is, both E and K are Γ-fields and the inclusion map $K \hookrightarrow E$ is Γ-equivariant, then we say that E/K is a *Galois extension of Γ-fields*. Note that, in such a case, for any $f \in G$ and $\sigma \in \Gamma$, the map ${}^{\sigma}f : E \to E$ given by ${}^{\sigma}f(x) = {}^{\sigma}(f({}^{\sigma^{-1}}x))$ defines a K-automorphism of E, that is, ${}^{\sigma}f$ is again an element of G. Therefore, the Galois group G is a Γ-group by the diagonal Γ-action $(\sigma, f) \mapsto {}^{\sigma}f$. We refer to G as the *Galois Γ-group* of the Galois extension of Γ-fields E/K.

From now on, we fix a finite Galois extension of Γ-fields E/K, with Galois Γ-group G.

The multiplicative group $E^{\times}$ is then both a Γ-module and a G-module with the natural actions of Γ and G on it. Moreover, the equalities ${}^{\sigma}(f(x)) = ({}^{\sigma}f)({}^{\sigma}x)$, $\sigma \in \Gamma$, $f \in G$, $x \in E^{\times}$, show that $E^{\times}$ is actually a $(G \rtimes \Gamma)$-module (see Lemma 1.1) and therefore the cohomology groups $H^n(G \rtimes \Gamma; E^{\times})$ and $H^n(G \rtimes \Gamma, \Gamma; E^{\times})$ are defined. Furthermore, the exact sequence (9) particularizes to the following one:

$$\cdots \to H^1\big(\Gamma; E^{\times}\big) \xrightarrow{\delta} H^2\big(G \rtimes \Gamma, \Gamma; E^{\times}\big) \xrightarrow{\text{in}} H^2\big(G \rtimes \Gamma; E^{\times}\big) \xrightarrow{\text{res}} H^2\big(\Gamma; E^{\times}\big). \tag{21}$$

Every $\omega \in \mathrm{Br}(E/K,\Gamma)$ has associated a cohomology class

$$\xi(\omega) \in H^2\big(G \rtimes \Gamma; E^{\times}\big), \tag{22}$$

related as "*the characteristic class of ω*," that is defined as follows. By Proposition 2.5, there exists a central simple (K,Γ)-algebra A in the $\sim_{\Gamma}$-class ω such that E is a (K,Γ)-subalgebra of A and $C_A(E) = E$. The Skolem–Noether theorem implies the existence of units $v_f \in A$, $f \in G$ ($v_1 = 1$), such that

$$f(a) \cdot v_f = v_f \cdot a, \quad \forall a \in E. \tag{23}$$

Each element v_f is obviously determined up to a factor that commutes with all elements of E and, since $C_A(E) = E$, this means up to a factor of $E^{\times}$. Then, for $f, g \in G$, $\sigma \in \Gamma$ and $a \in E$, the equalities

$$v_f \cdot v_g \cdot a = v_f \cdot g(a) \cdot v_g = f\big(g(a)\big) \cdot v_f \cdot v_g,$$
$${}^{\sigma}v_f \cdot a = {}^{\sigma}\big(v_f \cdot {}^{\sigma^{-1}}a\big) = {}^{\sigma}\big(f\big({}^{\sigma^{-1}}a\big) \cdot v_f\big) = \big({}^{\sigma}f\big)(a) \cdot {}^{\sigma}v_f$$

imply the existence of (unique) elements $\xi_A(f,g), \xi_A(f,\sigma) \in E^{\times}$ such that

$$v_f \cdot v_g = \xi_A(f,g) \cdot v_{fg}, \tag{24}$$
$${}^{\sigma}v_f = \xi_A(f,\sigma) \cdot v_{{}^{\sigma}f} \tag{25}$$

for all $f, g \in G$ and $\sigma \in \Gamma$.

In this way we obtain a reduced 2-cochain $\xi_A \in \bar{C}^2(G \rtimes \Gamma; E^\times)$

$$\xi_A : G^2 \cup G \times \Gamma \cup \Gamma^2 \to E^\times, \tag{26}$$

where $\xi_A|_{\Gamma^2} = 0$, which is actually a 2-cocycle, that is,

$$\xi_A \in \bar{Z}^2\big(G \rtimes \Gamma, \Gamma; E^\times\big) \subseteq \bar{Z}^2\big(G \rtimes \Gamma; E^\times\big).$$

In fact, it is well known that cocycle condition (11) is a consequence of the associative law $v_f \cdot (v_g \cdot v_h) = (v_f \cdot v_g) \cdot v_h$ in A. The cocycle condition (12) follows from the equality ${}^{\sigma}(v_f \cdot v_g) = {}^{\sigma}v_f \cdot {}^{\sigma}v_g$ since, on one hand,

$${}^{\sigma}(v_f \cdot v_g) = {}^{\sigma}\big(\xi_A(f,g) \cdot v_{fg}\big) = {}^{\sigma}\xi_A(f,g) \cdot \xi_A(fg,\sigma) \cdot v_{{}^{\sigma}f\,{}^{\sigma}g}$$

while, on the other hand,

$$\begin{aligned} {}^{\sigma}v_f \cdot {}^{\sigma}v_g &= \xi_A(f,\sigma) \cdot v_{{}^{\sigma}f} \cdot \xi_A(g,\sigma) \cdot v_{{}^{\sigma}g} \\ &= \xi_A(f,\sigma) \cdot \big({}^{\sigma}f\big)\big(\xi_A(g,\sigma)\big) \cdot \xi_A\big({}^{\sigma}f, {}^{\sigma}g\big) \cdot v_{{}^{\sigma}f\,{}^{\sigma}g}. \end{aligned}$$

Therefore, by comparison, (12) follows. Similarly, cocycle condition (13) follows from the equality ${}^{\sigma}({}^{\tau}v_f) = {}^{\sigma\tau}v_f$, and the last cocycle condition (14) is obviously satisfied.

The characteristic class of $\omega = [A] \in \mathrm{Br}(E/K, \Gamma)$ is then defined as

$$\xi(\omega) = [\xi_A] \in H^2\big(G \rtimes \Gamma; E^\times\big), \tag{27}$$

and our major result in this paper is stated in the theorem below.

Theorem 3.1. *For any finite-dimensional Galois extension of Γ-fields E/K, with Galois Γ-group G, the correspondence $\omega \mapsto \xi(\omega)$ establishes a well-defined homomorphism from the relative equivariant Brauer group* $\mathrm{Br}(E/K,\Gamma)$ *into the cohomology group $H^2(G \rtimes \Gamma; E^\times)$, and the sequence*

$$0 \to \mathrm{Br}(E/K,\Gamma) \xrightarrow{\xi} H^2\big(G \rtimes \Gamma; E^\times\big) \xrightarrow{\mathrm{res}} H^2\big(\Gamma; E^\times\big) \tag{28}$$

is exact.

Proof. The proof of this theorem consists in proving that the above sequence (28) occurs in a commutative diagram

$$\begin{tikzcd} H^1\big(\Gamma; E^\times\big) \arrow[r, "\delta"] & H^2\big(G \rtimes \Gamma; \Gamma; E^\times\big) \arrow[rr, "\mathrm{in}"] \arrow[dr, "\Delta"'] & & H^2\big(G \rtimes \Gamma; E^\times\big) \arrow[r, "\mathrm{res}"] & H^2\big(\Gamma; E^\times\big) \\ & & \mathrm{Br}(E/K,\Gamma) \arrow[ur, "\xi"'] & & \end{tikzcd} \tag{29}$$

where the sequence

$$H^1\big(\Gamma; E^\times\big) \xrightarrow{\delta} H^2\big(G \rtimes \Gamma, \Gamma; E^\times\big) \xrightarrow{\Delta} \mathrm{Br}(E/K, \Gamma) \longrightarrow 0 \tag{30}$$

is exact. Therefore, the exactness of (21) implies that ξ is well defined and that (28) is exact. The proof of the exactness of (30) can be naturally divided into two parts:

Part I: The epimorphism Δ

Let $\varphi \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$. We shall construct a (K, Γ)-algebra, denoted as $\Delta(\varphi)$, which we will call the *equivariant crossed product* of the Γ-group G and the Γ-field E relative to φ. Indeed, $\Delta(\varphi)$ is the ordinary crossed product algebra of the group G and the field E relative to the ordinary 2-cocycle $\varphi|_{G^2} : G^2 \to E^\times$; that is, the elements of the algebra $\Delta(\varphi)$ are the formal linear combinations $\sum_{f\in G} a_f \cdot u_f$, where a_f are elements of the field E and the u_f certain symbols indexed by G, with multiplication

$$(a \cdot u_f) \cdot (b \cdot u_g) = a \cdot f(b) \cdot \varphi(f, g) \cdot u_{fg}. \tag{31}$$

Thus, $\Delta(\varphi)$ is a central simple K-algebra that contains E, via the immersion $a \mapsto a \cdot u_1$, as a maximal commutative subalgebra. Moreover, Γ acts on $\Delta(\varphi)$ by the formula

$${}^\sigma(a \cdot u_f) = {}^\sigma a \cdot \varphi(f, \sigma) \cdot u_{{}^\sigma f}, \tag{32}$$

and $\Delta(\varphi)$ is actually a (K, Γ)-algebra since:

$$\begin{aligned}
{}^\sigma\big((a \cdot u_f) \cdot (b \cdot u_g)\big) &= {}^\sigma\big(a \cdot f(b) \cdot \varphi(f, g) \cdot u_{fg}\big) \\
&= {}^\sigma a \cdot {}^\sigma\big(f(b)\big) \cdot {}^\sigma\big(\varphi(f, g)\big) \cdot \varphi(fg, \sigma) \cdot u_{{}^\sigma(fg)} \\
&\overset{(12)}{=} {}^\sigma a \cdot \big({}^\sigma f\big)\big({}^\sigma b\big) \cdot \varphi(f, \sigma) \cdot \big({}^\sigma f\big)\big(\varphi(g, \sigma)\big) \cdot \varphi\big({}^\sigma f, {}^\sigma g\big) \cdot u_{{}^\sigma f\, {}^\sigma g} \\
&= \big({}^\sigma a \cdot \varphi(f, \sigma) \cdot u_{{}^\sigma f}\big) \cdot \big({}^\sigma b \cdot \varphi(g, \sigma) \cdot u_{{}^\sigma g}\big) \\
&= {}^\sigma(au_f) \cdot {}^\sigma(bu_g),
\end{aligned}$$

and

$${}^{\sigma\tau}(a \cdot u_f) = {}^{\sigma\tau}a \cdot \varphi(f, \sigma\tau) u_{{}^{\sigma\tau}f} \overset{(13)}{=} {}^\sigma\big({}^\tau a\big) \cdot {}^\sigma\big(\varphi(f, \tau)\big) \cdot \varphi\big({}^\tau f, \sigma\big) \cdot u_{{}^\sigma({}^\tau f)} = {}^\sigma\big({}^\tau(a \cdot u_f)\big).$$

The equalities ${}^\sigma(a \cdot u_1) = {}^\sigma a \cdot u_1$, $a \in E$, $\sigma \in \Gamma$, ensure that E is embedded in $\Delta(\varphi)$ as a (K, Γ)-subalgebra and, therefore, Proposition 2.5 implies that $[\Delta(\varphi)] \in \mathrm{Br}(E/K, \Gamma)$.

Lemma 3.2.

(i) *If $\varphi, \varphi' \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$ are two reduced relative to Γ 2-cocycles representing the same cohomology class in $H^2(G \rtimes \Gamma, \Gamma; E^\times)$, then the associated equivariant crossed product (K, Γ)-algebras $\Delta(\varphi)$ and $\Delta(\varphi')$ are isomorphic.*

(ii) *Every central simple (K,Γ)-algebra, containing E as a (K,Γ)-subalgebra commutative maximal, is equivariantly isomorphic to an equivariant crossed product $\Delta(\varphi)$ for some $\varphi \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$.*

Proof. (i) Let $\varphi, \varphi' \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$. If φ and φ' represent the same cohomology class in $H^2(G \rtimes \Gamma, \Gamma; E^\times)$, then they are made cohomologous by means of a normalized map $\psi : G \cup \Gamma \to E^\times$ such that $\xi_A|_\Gamma = 0$, that is, $\varphi(f,g) = \varphi'(f,g) \cdot f(\psi(g)) \cdot \psi(fg)^{-1} \cdot \psi(f)$ and $\varphi(f,\sigma) = \varphi'(f,\sigma) \cdot {}^\sigma\psi(f) \cdot \psi({}^\sigma f)^{-1}$, for all $f, g \in G$ and $\sigma \in \Gamma$. Therefore, the associated equivariant crossed product (K,Γ)-algebras $\Delta(\varphi)$ and $\Delta(\varphi')$ are made equivariantly isomorphic by the map $\Psi : a \cdot u_f \mapsto a \cdot \psi(f) \cdot u_f$. Indeed, it is known that Ψ establishes a K-algebra isomorphism, since ψ makes the ordinary 2-cocycles $\varphi|_{G^2}$ and $\varphi'|_{G^2}$ cohomologous. Furthermore, Ψ is Γ-equivariant since

$$\begin{aligned}\Psi\big({}^\sigma(au_f)\big) &= \Psi\big({}^\sigma a \cdot \varphi(f,\sigma) \cdot u_{{}^\sigma f}\big) = {}^\sigma a \cdot \varphi(f,\sigma) \cdot \psi\big({}^\sigma f\big) \cdot u_{{}^\sigma f} \\ &\overset{(16)}{=} {}^\sigma a \cdot {}^\sigma\psi(f) \cdot \varphi'(f,\sigma) \cdot u_{{}^\sigma f} = {}^\sigma\Psi(a \cdot u_f).\end{aligned}$$

(ii) Let A be any central simple (K,Γ)-algebra such that E is a (K,Γ)-subalgebra of A satisfying that $C_A(E) = E$. Then, let

$$\xi_A \in \bar{Z}^2\big(G \rtimes \Gamma, \Gamma; E^\times\big) \subseteq \bar{Z}^2\big(G \rtimes \Gamma; E^\times\big)$$

its characteristic 2-cocycle (26). Comparison of (23)–(25) with (31) and (32) gives that the mapping $\Phi : \sum_f a_f \cdot u_f \mapsto \sum_f a_f \cdot v_f$ defines a (K,Γ)-algebra homomorphism from the equivariant crossed product algebra $\Delta(\xi_A)$ to A. Since $\Delta(\xi_A)$ is simple, Φ is a monomorphism. Moreover, since Φ is E-linear and $[A:E] = [E:K] = |G|$, it follows that Φ is actually an isomorphism. $\square$

It follows from the above Lemma 3.2 that the equivariant crossed product construction induces a well-defined surjective map

$$\Delta : H^2\big(G \rtimes \Gamma, \Gamma; E^\times\big) \to \mathrm{Br}(E/K,\Gamma), \quad [\varphi] \mapsto \big[\Delta(\varphi)\big], \tag{33}$$

and a parallel proof to the usual one for the nonequivariant case proves that Δ is actually a homomorphism of groups: Given $\varphi, \varphi' \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$, then there is a (K,Γ)-algebra isomorphism $\Phi : \Delta(\varphi\varphi') \cong e \cdot (\Delta(\varphi) \otimes_K \Delta(\varphi')) \cdot e$, where $e \in E \otimes_K E$ is the separability idempotent of E, given by

$$\Phi : \sum_{f \in G} a_f \cdot u_f \mapsto \sum_{f \in G} e \cdot (a_f \cdot u_f \otimes u_f) \cdot e.$$

Then, by Lemma 2.4, $\Delta[\varphi\varphi'] = [\Delta(\varphi\varphi')] = [\Delta(\varphi) \otimes_K \Delta(\varphi')] = \Delta[\varphi]\Delta[\varphi']$.

Part II: The exactness of sequence (30)

We first prove that $\mathrm{Im}(\delta) \subseteq \mathrm{Ker}(\Delta)$:

Let $[\varphi] \in H^1(\Gamma; E^\times)$, where $\varphi : \Gamma \to E^\times$ is a derivation. Then $\Delta\delta[\varphi] = [\Delta(\delta_\varphi)]$, where δ_φ is the reduced 2-cocycle defined by (18). Observe that $\delta_\varphi(f, \sigma) = \varphi(\sigma) \cdot {}^\sigma(f(\varphi(\sigma^{-1})))$, since $\varphi(\sigma^{-1}) = ({}^{\sigma^{-1}}\varphi(\sigma))^{-1}$.

Write V for a 1-dimensional vector space over E with basis $\{v\}$. This can be given the structure of a (E, Γ)-vector space by the Γ-action

$$ {}^\sigma(a \cdot v) = \varphi(\sigma) \cdot {}^\sigma a \cdot v, $$

for $\sigma \in \Gamma$ and $a \in E$, since

$$ {}^\sigma\big({}^\tau(a \cdot v)\big) = {}^\sigma\big(\varphi(\tau) \cdot {}^\tau a \cdot v\big) = \varphi(\sigma) \cdot {}^\sigma\varphi(\tau) \cdot {}^{\sigma\tau}a \cdot v = \varphi(\sigma\tau) \cdot {}^{\sigma\tau}a \cdot v = {}^{\sigma\tau}(a \cdot v). $$

Of course, V is also a (K, Γ)-vector space. Let Φ be the K-linear map

$$ \Phi : \Delta(\delta_\varphi) \to \mathrm{End}_K(V), \quad \text{such that } \Phi(a \cdot u_f)(b \cdot v) = a \cdot f(b) \cdot v. \tag{34} $$

Φ is a K-algebra map since

$$ \begin{aligned} \Phi\big((a \cdot u_f)(b \cdot u_g)\big)(c \cdot v) &= \Phi\big(a \cdot f(b) \cdot u_{fg}\big)(c \cdot v) = a \cdot f(b) \cdot fg(c) \cdot v \\ &= \Phi(a \cdot u_f)\big(b \cdot g(c) \cdot v\big) = \Phi(a \cdot u_f)\Phi(b \cdot u_g)(c \cdot v), \end{aligned} $$

and, moreover, Φ is a Γ-equivariant map since

$$ \begin{aligned} \Phi\big({}^\sigma(a \cdot u_f)\big)(b \cdot v) &= \Phi\big({}^\sigma a \cdot \delta_\varphi(f, \sigma) \cdot u_{{}^\sigma f}\big)(b \cdot v) = {}^\sigma a \cdot \delta_\varphi(f, \sigma) \cdot \big({}^\sigma f\big)(b) \cdot v \\ &= {}^\sigma a \cdot \varphi(\sigma) \cdot {}^\sigma\big(f\big(\varphi\big(\sigma^{-1}\big)\big)\big) \cdot {}^\sigma\big(f\big({}^{\sigma^{-1}}b\big)\big) \cdot v \\ &= \varphi(\sigma) \cdot {}^\sigma\big[a \cdot f\big(\varphi\big(\sigma^{-1}\big) \cdot {}^{\sigma^{-1}}b\big)\big] \cdot v \\ &= {}^\sigma\big[a \cdot f\big(\varphi\big(\sigma^{-1}\big) \cdot {}^{\sigma^{-1}}b\big) \cdot v\big] = {}^\sigma\big[\Phi(a \cdot u_f)\big(\phi\big(\sigma^{-1}\big) \cdot {}^{\sigma^{-1}}b \cdot v\big)\big] \\ &= {}^\sigma\big[\Phi(a \cdot u_f) \cdot \big({}^{\sigma^{-1}}(b \cdot v)\big)\big] = {}^\sigma\big(\Phi(a \cdot u_f)\big)(b \cdot v). \end{aligned} $$

Therefore $\Phi : \Delta(\delta_\varphi) \to \mathrm{End}_K(V)$ is a (K, Γ)-algebra homomorphism. Since $\Delta(\delta_\varphi)$ is simple and $[\Delta(\delta_\varphi) : K] = [E : K]^2 = [V : K]^2 = [\mathrm{End}_K(V) : K]$, we conclude that Φ is actually a (K, Γ)-algebra isomorphism. Then, $[\Delta(\delta_\varphi)] = [\mathrm{End}_K(V)]$ is the identity element of the equivariant Brauer group $\mathrm{Br}(E/K, \Gamma)$ and therefore $\mathrm{Im}(\delta) \subseteq \mathrm{Ker}(\Delta)$.

Next we prove that $\mathrm{Ker}(\Delta) \subseteq \mathrm{Im}(\delta)$:

Let $[\varphi] \in \mathrm{Ker}(\delta)$, where $\varphi \in \bar{Z}^2(G \rtimes \Gamma, \Gamma; E^\times)$ is a relative to Γ reduced 2-cocycle. Then, by Proposition 2.3, there exists a Γ-equivariant K-algebra isomorphism $\theta : \Delta(\varphi) \cong \mathrm{End}_K(V)$ for some (K, Γ)-vector space V. We turn V into an E-vector space via θ, that is, by defining $a \cdot v = \theta(a \cdot u_1)(v)$ for $a \in E$ and $v \in V$. Since

$$\begin{aligned}
{}^{\sigma}(a\cdot v) &= {}^{\sigma}\big(\theta(a\cdot u_1)(v)\big) = {}^{\sigma}\big(\theta(a\cdot u_1)\big({}^{\sigma^{-1}}\big({}^{\sigma}v\big)\big)\big)\\
&= \big({}^{\sigma}\big(\theta(a\cdot u_1)\big)\big)\big({}^{\sigma}v\big) = \theta\big({}^{\sigma}a\cdot u_1\big)\big({}^{\sigma}v\big)\\
&= {}^{\sigma}a\cdot {}^{\sigma}v,
\end{aligned}$$

we see that V is actually an (E,Γ)-vector space.

Observe now that $[V:E]=1$ because $[\Delta(\varphi):K]=[E:K]^2$ while $[\mathrm{End}_K(V):K]=[V:K]^2=[V:E]^2[E:K]^2$. Fix $v\in V$ any nonzero vector. Then, for any $\sigma\in\Gamma$, we can write ${}^{\sigma}v=\phi(\sigma)\cdot v$ for some (unique) $\phi(\sigma)\in E^{\times}$. Since ${}^{\sigma\tau}v=\phi(\sigma\tau)\cdot v$ while ${}^{\sigma}({}^{\tau}v)={}^{\sigma}(\phi(\tau)\cdot v)={}^{\sigma}\phi(\tau)\cdot{}^{\sigma}v={}^{\sigma}\phi(\tau)\cdot\phi(\sigma)\cdot v$, we have $\phi(\sigma\tau)={}^{\sigma}\phi(\tau)\cdot\phi(\sigma)$ for all $\sigma,\tau\in\Gamma$. Therefore, $\phi:\Gamma\to E^{\times}$ is a derivation.

By composing θ with the inverse of the (K,Γ)-algebra isomorphism (34), $\Phi:\Delta(\delta_\phi)\cong\mathrm{End}_K(V)$, we get a (K,Γ)-algebra isomorphism $\Psi:\Delta(\varphi)\cong\Delta(\delta_\phi)$, that is also an E-linear map. For every $f\in G$, in both algebras $\Delta(\delta_\phi)$ and $\Delta(\varphi)$ the vector u_f generates the vector subspace over E consisting of all x such that $x\cdot a=f(a)\cdot x$ for all $a\in E$. Therefore, we can write $\Psi(u_f)=\psi(f)\cdot u_f$ for some (unique) $\psi(f)\in E^{\times}$. Thus, we have a normalized map $\psi:G\cup\Gamma\to E^{\times}$, such that $\psi|_\Gamma=0$ and $\Psi(a\cdot u_f)=a\cdot\psi(f)\cdot u_f$ for any $a\in E$ and $f\in G$.

Now, since Ψ is a homomorphism of algebras and

$$\Psi(u_f\cdot u_g)=\Psi\big(\varphi(f,g)\cdot u_{fg}\big)=\varphi(f,g)\cdot\psi(fg)\cdot u_{fg},$$

whereas

$$\Psi(u_f)\cdot\Psi(u_g)=\psi(f)\cdot u_f\cdot\psi(g)\cdot u_g=\psi(f)\cdot f\big(\psi(g)\big)\cdot u_{fg},$$

we have

$$\varphi(f,g)=\psi(f)\cdot f\big(\psi(g)\big)\cdot\psi(fg)^{-1}=(\partial\psi)(f,g)=(\delta_\phi\cdot\partial\psi)(f,g),$$

for all $f,g\in G$.

Furthermore, since Ψ is Γ-equivariant and

$$\Psi\big({}^{\sigma}u_f\big)=\Psi\big(\varphi(f,\sigma)\cdot u_{{}^{\sigma}f}\big)=\varphi(f,\sigma)\cdot\psi\big({}^{\sigma}f\big)\cdot u_{{}^{\sigma}f},$$

while

$${}^{\sigma}\Psi(u_f)={}^{\sigma}\big(\psi(f)\cdot u_f\big)={}^{\sigma}\big(\psi(f)\big)\cdot\delta_\phi(f,\sigma)\cdot u_{{}^{\sigma}f},$$

we have

$$\varphi(f,\sigma)=\delta_\phi(f,\sigma)\cdot{}^{\sigma}\big(\psi(f)\big)\cdot\psi\big({}^{\sigma}f\big)^{-1}=(\delta_\phi\cdot\partial\psi)(f,\sigma),$$

for all $f\in G$ and $\sigma\in\Gamma$. Therefore, $[\varphi]=[\delta_\phi\cdot\partial\psi]=[\delta_\phi]=\delta[\phi]$ and therefore $\mathrm{Ker}(\Delta)\subseteq\mathrm{Im}(\delta)$.

The proof of Theorem 3.1 is now finished. $\square$

The epimorphism $\Delta : H^2(G \rtimes \Gamma, \Gamma; E^\times) \twoheadrightarrow \mathrm{Br}(E/K, \Gamma)$ has an identifiable kernel: By the commutativity of (29), the kernel of Δ is the same as the kernel of the homomorphism $H^2(G \rtimes \Gamma, \Gamma; E^\times) \to H^2(G \rtimes \Gamma; E^\times)$. Then we can take into account exact sequence (9), and, in particular, the exact sequence

$$H^1(G \rtimes \Gamma; E^\times) \to H^1(\Gamma; E^\times) \to H^2(G \rtimes \Gamma, \Gamma; E^\times) \to H^2(G \rtimes \Gamma; E^\times).$$

Moreover, from the Hochschild–Serre spectral sequence [7, Theorem 2], associated to the group extension $1 \to G \to G \rtimes \Gamma \to \Gamma \to 1$ and the $(G \rtimes \Gamma)$-module $E^\times$, we have the exact sequence

$$0 \to H^1(\Gamma; K^\times) \to H^1(G \rtimes \Gamma; E^\times) \to H^1(G; E^\times),$$

and using that $H^1(G; E^\times) = 0$, that is, Hilbert's Theorem 90, we deduce that $H^1(\Gamma; K^\times) \cong H^1(G \rtimes \Gamma; E^\times)$. Therefore, exact sequence (30) implies the exactness of the sequence

$$H^1(\Gamma; K^\times) \to H^1(\Gamma; E^\times) \to H^2(G \rtimes \Gamma, \Gamma; E^\times) \xrightarrow{\Delta} \mathrm{Br}(E/K, \Gamma) \to 0,$$

whence it follows that

Proposition 3.3. *Let E/K be a finite Galois extension of Γ-fields with Galois Γ-group G. Then the kernel of the epimorphism*

$$H^2(G \rtimes \Gamma, \Gamma; E^\times) \xrightarrow{\Delta} \mathrm{Br}(E/K, \Gamma) \to 0$$

is isomorphic to the cokernel of the canonical homomorphism

$$H^1(\Gamma; K^\times) \to H^1(\Gamma; E^\times).$$

This fact implicitly states that if the map $H^1(\Gamma; K^\times) \to H^1(\Gamma; E^\times)$ is not surjective, then *there are central simple (K, Γ)-algebras split by the Γ-field E which have the same dimension and define the same element in* $\mathrm{Br}(K, \Gamma)$, *but they are not Γ-isomorphic*. We stress this observation because it is unusual. Indeed, it is easy to show examples where that fact holds. Suppose that Γ acts trivially both on K and E; then, $\mathrm{End}_K(E)$, with the trivial Γ-action, is a central simple (K, Γ)-algebra that represents the trivial element in $\mathrm{Br}(E/K, \Gamma)$. Suppose also that the homomorphism $\mathrm{Hom}(\Gamma; K^\times) \to \mathrm{Hom}(\Gamma; E^\times)$ is not surjective, so that there is a homomorphism, say $\phi : \Gamma \to E^\times$, such that $\phi(\sigma) \notin K$ for some $\sigma \in \Gamma$. Then, E can be given the structure of (K, Γ)-vector space by the Γ-action ${}^\sigma e = \phi(\sigma) \cdot e$, $\sigma \in \Gamma$, $e \in E$. Denote this (K, Γ)-space by ${}_\phi E$. Then $\mathrm{End}_K({}_\phi E)$ is a central simple (K, Γ)-algebra also representing the trivial element in $\mathrm{Br}(E/K, \Gamma)$. Note that $\mathrm{End}_K({}_\phi E)$ is actually the same K-algebra as $\mathrm{End}_K(E)$ but with Γ-action ${}^\sigma f = \ell_{\phi(\sigma)} f \ell_{\phi(\sigma)^{-1}}$, where ℓ_e is left multiplication by e, that is, $({}^\sigma f)(e) = \phi(\sigma) f(\phi(\sigma^{-1})e)$. Now, since $\phi(\sigma) \notin K$ and K is the center of $\mathrm{End}_K(E)$, then ${}^\sigma f \neq f$ for some $f \in \mathrm{End}_K({}_\phi E)$ and therefore $\mathrm{End}_K(E)$ and $\mathrm{End}_K({}_\phi E)$ are not Γ-isomorphic, although they are the same K-algebra and both represent the same element in $\mathrm{Br}(K, \Gamma)$.

As a final comment, we shall stress the following consequence of Theorem 3.1, which is due to the referee:

Proposition 3.4. *Let E/K be a finite Galois extension of Γ-fields with Galois Γ-group G. Then the kernel of the "forgetting Γ" homomorphism*

$$\mathrm{Br}(E/K,\Gamma) \to \mathrm{Br}(E/K)$$

is isomorphic to the kernel of the canonical homomorphism

$$H^2\big(\Gamma; K^\times\big) \to H^2\big(\Gamma; E^\times\big).$$

Proof. By using that $H^1(G; E^\times) = 0$, the Hochschild–Serre spectral sequence associated to the semidirect product extension gives the exact sequence [7, Theorem 2]

$$0 \to H^2\big(\Gamma; K^\times\big) \to H^2\big(G \rtimes \Gamma; E^\times\big) \to H^2\big(G; E^\times\big).$$

Let Ker denote the kernel of the homomorphism $\mathrm{Br}(E/K,\Gamma) \to \mathrm{Br}(E/K)$. Then, there is a unique, induced by ξ, homomorphism $\mathrm{Ker} \dashrightarrow H^2(\Gamma; K^\times)$ making the following diagram commutative:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Ker} \arrow[r, dashed] \arrow[d] & H^2\big(\Gamma; K^\times\big) \arrow[r] \arrow[d] & H^2\big(\Gamma; E^\times\big) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Br}(E/K,\Gamma) \arrow[r, "\xi"] \arrow[d] & H^2\big(G \rtimes \Gamma; E^\times\big) \arrow[r] \arrow[d] & H^2\big(\Gamma; E^\times\big) \\
 & \mathrm{Br}(E/K) \arrow[r, "\sim"] & H^2\big(G; E^\times\big) &
\end{tikzcd}$$

from which the exactness of the sequence below follows:

$$0 \to \mathrm{Ker} \dashrightarrow H^2\big(\Gamma; K^\times\big) \longrightarrow H^2\big(\Gamma; E^\times\big). \qquad \square$$

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